NLPP - Problems on Multivariable optimization with Inequality Constraints: [Constrained optimization]

Solve the Non Linear Programming Problem (NLPP)

Problem: Minimize $f(x_1, x_2) = x_1^2 + x_2^2 - 14x_1 - 6x_2$

$$g_1(x_1, x_2) = x_1 + x_2 - 2 \leq 0.$$

$$g_2(x_1, x_2) = x_1 + 2x_2 - 3 \leq 0$$

Solution: Case(i) Assume both Constraints g_1, g_2 are active

$$L = (x_1^2 + x_2^2 - 14x_1 - 6x_2) + \lambda_1(x_1 + x_2 - 2) + \lambda_2(x_1 + 2x_2 - 3)$$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 14 + \lambda_1 + \lambda_2 = 0$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 6 + \lambda_1 + 2\lambda_2 = 0$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 - 2 = 0$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow x_1 + 2x_2 - 3 = 0$$

$$2x_1 + 0x_2 + 1.\lambda_1 + 1.\lambda_2 = 14.$$

$$0x_1 + 2x_2 + 1.\lambda_1 + 2\lambda_2 = 6.$$

$$1.x_1 + 1.x_2 + 0\lambda_1 + 0\lambda_2 = 2.$$

$$1.x_1 + 2.x_2 + 0\lambda_1 + 0\lambda_2 = 3$$

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 14 \\ 6 \\ 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 14 \\ 0 & 2 & 1 & 2 & 6 \\ 1 & 1 & 0 & 0 & 2 \\ 1 & 2 & 0 & 0 & 3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\sim \left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 14 \\ 0 & 2 & 1 & 2 & 6 \\ 0 & 2 & -1 & -1 & -10 \\ 0 & 4 & -1 & -1 & -8 \end{array}\right] \begin{array}{l} R_1' = R_1 \\ R_2' = R_2 \\ R_3' = 2R_3 - R_1 \\ R_4' = 2R_4 - R_1 \end{array}$$

$6 - 14 = -8$

$$\sim \left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 14 \\ 0 & 2 & 1 & 2 & 6 \\ 0 & 0 & -2 & -3 & -16 \\ 0 & 0 & -3 & -5 & -20 \end{array}\right] \begin{array}{l} R_1'' = R_1' \\ R_2'' = R_2' \\ R_3'' = R_3' - R_2' \\ R_4'' = R_4' - 2R_2' \end{array}$$

$$\begin{array}{cccc} x_1 & x_2 & \lambda_1 & \lambda_2 \end{array}$$

$$\sim \left[\begin{array}{ccccc} 2 & 0 & 1 & 1 & 14 \\ 0 & 2 & 1 & 2 & 6 \\ 0 & 0 & -2 & -3 & -16 \\ 0 & 0 & 0 & -1 & 8 \end{array}\right] \begin{array}{l} R_1''' = R_1'' \\ R_2''' = R_2'' \\ R_3''' = R_3'' \\ R_4''' = 2R_4'' - 3R_3'' \end{array}$$

$-8 - 12$

From backward substitution, we get

$-\lambda_2 = 8$

$\boxed{\lambda_2 = -8}$

$-10 + 9$

-40

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$-2\lambda_1 - 3\lambda_2 = -16$

$2\lambda_1 + 3\lambda_2 = 16$

$2\lambda_1 + 3(-8) = 16$

$2\lambda_1 - 24 = 16$

$2\lambda_1 = 40$

$\boxed{\lambda_1 = 20}$

$2x_2 + \lambda_1 + 2\lambda_2 = 6$

$\Rightarrow 2x_2 + 20 + 2(-8) = 6$

$\Rightarrow 2x_2 + 20 - 16 = 6$

$\Rightarrow 2x_2 = 2$

$\boxed{x_2 = 1}$

$2x_1 + \lambda_1 + \lambda_2 = 14$

$\Rightarrow 2x_1 + 20 + (-8) = 14$

$\Rightarrow 2x_1 + 12 = 14$

$\Rightarrow 2x_1 = 2$

$\boxed{x_1 = 1}$

$\therefore$ Critical points are $(\cancel{x_1, x_2},\ \lambda_1 = 20,\ \lambda_2 = -8$
$x_1 = 1,\ x_2 = 1,$

$f(x_1, x_2) = f(1,1) = 1 + 1 - 14 - 6 = -18$

$g_1(x_1, x_2) = 1 + 1 - 2 = 0$

$g_2(x_1, x_2 = 1 + 2 - 3 = 0$

$(1,1)$ satisfies both constraints g_1 and g_2.

$\therefore$ Min $f = -18$, and occurs at $x_1 = 1,\ x_2 = 1$

Hower the solution is not optimum

($\because \lambda_2 = -8$, should not be negative)

Case (ii)

Omiting the both constraints g_1 and g_2

$$f(x_1, x_2) = x_1^2 + x_2^2 - 14x_1 - 6x_2 \quad [g_1 \text{ and } g_2 \text{ are not active}]$$

we get $L = x_1^2 + x_2^2 - 14x_1 - 6x_2$

Necessary conditions are $\frac{\partial L}{\partial x_1} = 0$ and $\frac{\partial L}{\partial x_2} = 0$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 14 = 0 \Rightarrow x_1 = 7$.

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 6 = 0 \Rightarrow x_2 = 3$.

$g_1(7,3) = x_1 + x_2 - 2 = 7 + 3 - 2 = 8 > 0$

$g_2(7,3) = x_1 + 2x_2 - 3 = 7 + 2(3) - 3 = 7 + 6 - 3 = 10 > 0$.

But both $g_1 \leq 0$; and $g_2 \leq 0$.

$\therefore$ $(7,3)$ is not solution.

Case (iii) $\rightarrow$ $\boxed{\text{Let } g_1 \text{ be active, } g_2 \text{ Not active}}$

$\therefore L = x_1^2 + x_2^2 - 14x_1 - 6x_2 + \lambda_1 (x_1 + x_2 - 2)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 14 + \lambda_1 = 0 \rightarrow (1)$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 6 + \lambda_1 = 0 \rightarrow (2)$

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x_1 + x_2 - 2 = 0 \rightarrow (3)$

(1) $-$ (2) $\Rightarrow 2x_1 - 2x_2 - 8 = 0$

and $x_1 + x_2 - 2 = 0$

$\Rightarrow$

$$\begin{aligned} x_1 - x_2 &= 4 \\ (-)\quad x_1 + x_2 &= 2 \\ \hline -2x_2 &= -6 \end{aligned}$$

$\Rightarrow \boxed{x_2 = 3}$

Sub. $x_1 = 3$ in

$$x_1 + x_2 = 2$$

$$x_2 = 2 - 3 = -1$$

$$\boxed{x_2 = -1}$$

Sub. $x_1 = 3$ in (1)

$$2x_1 + \lambda_1 = 14$$

$$6 + \lambda_1 = 14$$

$$\boxed{\lambda_1 = 8}$$

$\therefore \quad x_1 = 3, \quad x_2 = -1, \quad \lambda_1 = 8 > 0$

$$g_1(x_1, x_2) = x_1 + x_2 - 3$$

$$g_1(3, -1) = 3 - 1 - 3 = -1 \underline{\underline{< 0}}$$

$$g_2(3, -1) = 3 + 2(-1) - 3 = 3 - 2 - 3 = \underline{\underline{-2 < 0}}$$

$$f(3, -1) = 3^2 + (-1)^2 - 14(3) - 6(-1)$$

$$= 9 + 1 - 42 + 6$$

$$= 16 - 42$$

$f(3, -1) = -26$ (the solution is optimal)

<u>Case (iv)</u> Assume g_2 is active, g_1 inactive

$\therefore \quad L = x_1^2 + x_2^2 - 14x_1 - 6x_2 + \lambda_2(x_1 + 2x_2 - 3)$

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - 14 + \lambda_2 = 0 \rightarrow (1)$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 2x_2 - 6 + 2\lambda_2 = 0 \rightarrow (2)$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow x_1 + 2x_2 - 3 = 0 \rightarrow (3)$$

Solving:

$\Rightarrow$ $2x_1 + \lambda_2 = 14 \rightarrow (1)$

$2x_2 + 2\lambda_2 = 6 \rightarrow (2)$

$\cancel{x_1 +}$

$(1) \times 2 \Rightarrow \quad 4x_1 + 2\lambda_2 = 28$

$2x_2 + 2\lambda_2 = 6$

$\cancel{2x_2}$

$4x_1 - 2x_2 = 22$

$\Rightarrow \quad 2x_1 - x_2 = 11.$

Solving, $\left.\begin{array}{l} 2x_1 - x_2 = 11 \\ x_1 + 2x_2 = 3 \end{array}\right\}$

$2x_1 - x_2 = 11$

$\underset{(-)}{2x_1} + \underset{(-)}{4x_2} = \underset{(-)}{6}$

$-5x_2 = 5$

$x_2 = -1$

$\boxed{x_2 = -1}$

Sub. $x_2 = -1$ in. $2x_1 - x_2 = 11$

$2x_1 - (-1) = 11$

$2x_1 = 10$

$\boxed{x_1 = 5}$

Put $x_1 = 5$ in. $2x_1 + \lambda_2 = 14$.

$10 + \lambda_2 = 14$

$\boxed{\lambda_2 = 4}$

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Solving:

$\Rightarrow \quad 2x_1 + \lambda_2 = 14 \rightarrow (1)$

$2x_2 + 2\lambda_2 = 6 \rightarrow (2)$

$(1) \times 2 \Rightarrow \quad 4x_1 + 2\lambda_2 = 28$

$2x_2 + 2\lambda_2 = 6$

$4x_1 - 2x_2 = 22$

$\Rightarrow \quad 2x_1 - x_2 = 11$

Solving,

$2x_1 - x_2 = 11$

$x_1 + 2x_2 = 3$

$2x_1 - x_2 = 11$

$2x_1 + 4x_2 = 6$

$(-) \quad (-) \quad (-)$

$-5x_2 = 5$

$x_2 = -1$

$\boxed{x_2 = -1}$

Sub. $x_2 = -1$ in. $2x_1 - x_2 = 11$

$2x_1 - (-1) = 11$

$2x_1 = 10$

$\boxed{x_1 = 5}$

Put $x_1 = 5$ in. $2x_1 + \lambda_2 = 14$

$10 + \lambda_2 = 14$

$\boxed{\lambda_2 = 4}$

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So $x_1 = 5,\ x_2 = -1,\ \lambda_2 = 4 > 0$

$\therefore$ ~~$g_1($~~ $\therefore$ Critical Points are (x_1, x_2, λ_2)

$(5, -1, 4)$

$g_1(5,-1) = x_1 + x_2 - 2$

$= 5 - 1 - 2$

$= 2 > 0$ ×

$g_2(5,-1) = 5 + 2(-1) - 3$

$= 5 - 2 - 3$

$g_2(5,-1) = 0$ ✓

$f(5,-1) = 26 + 1 - 14(5) - 6(-1)$

$= 27 - 70 + 6$

$= 33 - 70$

$= -37$ (not optimal)

$\therefore$ Optimum Solution is

$\boxed{x_1 = 3,\ x_2 = -1 \quad \text{Min} f = -26.}$

Problem 2: ~~KKT Example~~: Solve the NLPP

$$U = xy$$

s.t $x + y \le 50$ [income constraint]

$2x + y \le 60$ [coupen Constraint]

Sol:

$$L = xy + \lambda_1 (x + y - 50) + \lambda_2 (2x + y - 60) \rightarrow (1)$$

$\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial \lambda_1} = 0$, $\frac{\partial L}{\partial \lambda_2} = 0$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow 1(y) + \lambda_1 (1 + 0 - 0) + \lambda_2 (2(1) + 0 - 0) = 0.$$

$$\Rightarrow y + \lambda_1 + 2\lambda_2 = 0 \rightarrow (2)$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow x(1) + \lambda_1 (0 + 1 - 0) + \lambda_2 (0 + 1 - 0) = 0$$

$$\Rightarrow x + \lambda_1 + \lambda_2 = 0 \rightarrow (3)$$

$$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow 0 + 1(x + y - 50) + 0 = 0$$

$$x + y - 50 = 0 \rightarrow (4)$$

$$\frac{\partial L}{\partial \lambda_2} = 0 \Rightarrow \quad 0 + 0 + 1(2x + y - 60) = 0$$

$$2x + y - 60 = 0 \rightarrow (5)$$

Solving (4) and (5), we get

$$x + y = 50$$
$$2x + y = 60$$
$$(-) \quad (-) \quad (-)$$

$$-x = -10$$

$$\boxed{x = 10}$$

$\therefore$ $y = 50 - x = 50 - 10 = 40$

$$\boxed{y = 40}$$

Substitute x and y values in equations (2) & (3)

we get

$$40 + \lambda_1 + 2\lambda_2 = 0$$
$$10 + \lambda_1 + \lambda_2 = 0$$

$$\Rightarrow \quad \lambda_1 + 2\lambda_2 = -40$$
$$\lambda_1 + \lambda_2 = -10$$
$$(-) \quad (-) \quad (+)$$

$$\lambda_2 = -30$$

Sub. $\lambda_2 = -30$ in $\lambda_1 + 2\lambda_2 = -40$

we get $\lambda_1 - 60 = -40$

$$\boxed{\lambda_1 = 20}$$

$\therefore$ $x = 10$, $y = 40$, $\lambda_1 = 20$, $\lambda_2 = -30$

~~But $\lambda_2 = -30$ should~~

Here $g_1(x, y) = x + y - 50 = 10 + 40 - 50$

$= \underline{\underline{0}}$ ✓

$g_2(x,y) = 2x+y-60 = 2(10)+40-60$
$= 0$ ✓

But $\lambda_2 = -30$, should not be negative.

Case(i) Omitting the constraints ~~g~~ g_1 and g_2.

$L = xy$.

$\frac{\partial L}{\partial x} = y$ $\quad \frac{\partial L}{\partial x} = 0 \Rightarrow y = 0$

$\frac{\partial L}{\partial y} = x$ $\quad \frac{\partial L}{\partial y} = 0 \Rightarrow x = 0$.

$g_1(0,0) = 0+0-50 = -50 \leq 0$ ✓

$g_2(0,0) = 2(0)+0-60 = -60 \leq 0$ ✓

~~Bu~~ ~~Both~~ The point $(0,0)$ satisfies both g_1 and g_2

$\therefore$ ~~$U(0,0) = 0 \times 0 = 0$~~.

$\therefore$ ~~$U(x,y) = U(0,0)$~~ $=$

$U(x,y) = xy$ becomes

$\boxed{U(0,0) = 0}$

Case(ii) Assume that g_1 is active.

$\therefore L = xy + \lambda_1(x+y-50)$

$\frac{\partial L}{\partial x} = 0 \Rightarrow y + \lambda_1 = 0$

$\frac{\partial L}{\partial y} = 0 \Rightarrow x + \lambda_1 = 0$

$\frac{\partial L}{\partial \lambda_1} = 0 \Rightarrow x+y-50 = 0$

Adding first two equations we get

$x+y+2\lambda_1 = 0$ and

Solving we get $x+y-50 = 0$

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$$x+y+2\lambda_1=0$$
$$x+y-50=0$$
$$(-)\quad(-)\quad(+)$$

$$2\lambda_1+50=0$$

$$\boxed{\lambda_1=-25}$$

Since $\lambda_2=-25<0$, the solution is not optimum.

<u>Case(iii)</u> Assume g_2 is active

$$\therefore L = xy+\lambda_2(2x+y-60)$$

$$\frac{\partial L}{\partial x}=0 \Rightarrow y+2\lambda_2=0$$

$$\frac{\partial L}{\partial y}=0 \Rightarrow x+\lambda_2=0$$

$$\frac{\partial L}{\partial \lambda_2}=0 \Rightarrow 2x+y-6=0$$

From first two equations we get

$$x+y+3\lambda_2=0$$

and

$$2x+y-6=0$$

solving

$$(-)\quad(-)\quad(+)$$

$$-x+3\lambda_2+6=0$$

~~From eqn~~ But $x+\lambda_2=0$

$$\Rightarrow x=-\lambda_2$$

Subs in above equation

$$\lambda_2+3\lambda_2=-6$$

$$4\lambda_2=-6$$

$$\boxed{\lambda_2=-3/2}<0$$

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which should not be negative.

$\therefore$ Optimum solution is

$\boxed{x = 0,\ y = 0 \text{ and } u = 0}$